QP CODE: 23144638

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, NOVEMBER 2023

Third Semester

Faculty of Science

CORE - ME010304 - FUNCTIONAL ANALYSIS

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

EC811C12

Time: 3 Hours | Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. *Define a complete metric space. Give an example of incomplete space.*
2. Define Hamel basis. Give an example.
3. Prove that a linear operator preserves linear dependence.
4. What is meant by second algebraic dual space of a vector space and define algebraically reflexivity of a vector space.
5. Find the dual basis of the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ for R^3.
6. Define a convex subset of a vectorspace. Give an example.
7. Define Fourier series and Fourier coefficients.
8. Define a Sesquilinear form.
9. *Define partially ordered set. State Zorn's Lemma.*
10. *Define adjoint operator of bounded linear operator T from a normed space X into a normed space Y. If $S,T \in B(X,Y)$, prove that* $(S+T)^{\times} = S^{\times} + T^{\times}$

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Show that in a normed space X, vector addition and scalar multiplication are continuous operations with respect to the norm.
12. Prove that every finite dimensional subspace Y of a normed space X is closed in X.

13. Define a bounded linear operator on a normed space and prove that $\|T\| = sup\{\|Tx\|/x \in D(T), \|x\| = 1\}$. Also show that this alternate formula for norm satisfies all the conditions of a norm.

14. Let $T : X \to Y$ be a linear operator where X and Y are Normed spaces. Then prove that T is continuous if and only if T is bounded.

15. Show that $|\langle x, y\rangle| \leq \|x\|\|y\|$ in an inner product space.

16. Explain Gram-Schmidt process for orthonormalizing a linearly independent sequence in an inner product space.

17. *Let H_1 and H_2 are Hilbert spaces and $S, T \in B(H_1, H_2)$ then prove that*

   1. $(T^*)^* = T$
   2. $||TT^*|| = ||T^*T|| = ||T||^2$
   3. $T^*T = 0 \iff T = 0$
   4. $(ST)^* = T^*S^*$ (assuming $H_1 = H_2$)

18. *Let U be a unitary operator on a Hilbert space H, prove that*

   1. *U is isometric*
   2. $|||U|| = 1$
   3. U^{-1} *is unitary*
   4. *U is normal*

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any ***two*** *questions.*

Weight ***5*** *each.*

19. (i) State and prove Riesz's Lemma.
(ii) If a normed space X has the property that the closed unit ball $M = \{x/||x|| \leq 1\}$ is compact, then prove that X is finite dimensional.

20. i)Show that the dual space of R^n is R^n
ii) Show that dual space X^l of a normed space X is a Banach space.

21. Prove that two Hilbert spaces H and $\tilde{H}$, both real or complex, are isomorphic if and only if they have the same Hilbert dimension.

22.
   1. *State and prove Hahn-Banach theorem for normed spaces.*
   2. *Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then prove that there exists a bounded linear functional f on X such that $||f|| = 1$ and $f(x_0) = ||x_0||$.*

(2×5=10 weightage)