QP CODE: 24018747

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Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , APRIL 2024

Second Semester

CORE - ME010205 - MEASURE AND INTEGRATION

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

7B2338B3

Time: 3 Hours   Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define Lebesgue outer measure. Prove that the outer measure is monotone.
2. Prove that the Lebesgue measurable sets possess excision property.
3.
   1. When will you say a property holds almost everyehere on a measurable set E?
   2. Let $\{E_k\}_{k=1}^{\infty}$ is a countable collection of Lebesgue measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then prove that almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.
4. Let f and g are Lebesgue measurable functions that are finite a.e. on E. Prove that for any a and b , $af + bg$ is Lebesgue measurable on$E$.
5. Let $\{f_n\}$ be a sequence of bounded Lebesgue measureable functions on a set of finite measure$E$. Prove that if $\{f_n\} \to f$ uniformly on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$
6. State and prove Beppo Levi's Lemma for non negative Lebesgue measurable functions on E.
7. Define positive sets, negative sets and null sets with respect to a signed measure.
8. Define measurable functions on a measurable space $(X, \mathcal{M})$.
9. *Let $(X, \mathcal{M}, \mu)$ be a measure space and ψ be nonnegative simple functions on X. If* $X_0 \subseteq X$ is measurable and $\mu(X - X_0) = 0$, then prove that $\int_X \psi \, d\mu = \int_{X_0} \psi \, d\mu$
10. Let X be a compact topological space and $\mathcal{M}$ a σ-algebra of subsets of X that contains topology on X. If f is a continuous real-valued function on X and $(X, \mathcal{M}, \mu)$ is a finite measure space, then prove that f is integrable over X with respect μ.

(8×1=8 weightage)

Turn Over

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Prove that every interval is measurable.

12. Prove that there is a closed, uncountable set of measure zero.

13. State and prove the necessary and sufficient condition for Lebesgue measurability of an extended real valued function f on a measurable set E, using a sequence of simple functions $\{\phi_n\}$ on E.

14. Define Lebesgue integral of simple functions defined on a set of finite measure E. Also Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of Lebesgue measurable sets of a set of finite measure E. Prove that If a_i be a real number and for $1 \leq i \leq n$, $\phi = \Sigma_{i=1}^{n} a_i \chi_{E_i}$ on E, then $\int_E \phi = \Sigma_{i=1}^{n} a_i m(E_i)$.

15. State and prove the continuity properties of general measure.

16. If E_1 and E_2 are measurable sets, prove that $E_1 \cup E_2$ is measurable.

17. State and prove the Beppo Levi's Lemma of integration with respect to general measure.

18. *Let* $E \subset X \times Y$ be an $R_{\sigma\delta}$ set for which $(\mu \times \nu)E < \infty$. Then prove that $(\mu \times \nu)E = \int_X \nu(E_x)\, d\mu(x)$

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Prove that the collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains the σ-algebra $\mathcal{B}$ of Borel sets.

20. Prove that Lebesgue integration of bounded Lebesgue measurable functions on sets of finite measure is 1) Linear 2) Monotonic 3) Additive over domains of integration

21. Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Prove that there exists a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$ Also prove that the pair $\{A, B\}$ is unique except for null sets.

22. (a) When two measures on a measurable space are said to be mutually singular? Give an example of two mutually singular measures.
    (b) State and prove Lebesgue Decomposition theorem.

(2×5=10 weightage)