**F 5066** (Pages : 3) Reg. No..............................

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# M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2016

## Third Semester

Faculty of Science

Branch I (A)—Mathematics

## MT 03 C13—DIFFERENTIAL GEOMETRY

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

### Part A

*Answer any* **five** *questions.*
*Each question has weight* 1.

1. Find the gradient field of $X(x_1, x_2) = (x_2, x_1)$.
2. Define (i) Regular point of a smooth function ; (ii) Tangent space.
3. Define a geodesic. Show that geodesics have constant speed.
4. Prove that if X and Y are two parallel vector fields along $\alpha$, $X \cdot Y$ is constant along $\alpha$.
5. Compute $\nabla_v f$ where $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = 2x_1^2 + 3x_2^2$, where $v = (1, 0, 2, 1)$.
6. Define a differential 1-form. How will you define the sum of two1-forms.
7. Define a parametrized $n$-surface. Write the map which represent the parametrized torus in $\mathbb{R}^4$.
8. State inverse function theorem.

$(5 \times 1 = 5)$

### Part B

*Answer any* **five** *questions.*
*Each question has weight* 2.

9. Sketch the level curves and graph of $f(x_1, x_2) = x_1^2 - x_2^2$.
10. Let $S \subset \mathbb{R}^{n+1}$ be a connected $n$-surface in $\mathbb{R}^{n+1}$. Prove that there exist on S exactly two smooth unit normal vector fields $N_1$ and $N_2$ and $N_2(p) = -N_1(p)$ for all $p \in S$.

Turn over

11. Describe the spherical image when $n = 1$ and $n = 2$ of the surface $-x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 0, x_1 > 0$, oriented by $\nabla f/\|\nabla f\|$.

12. Let S be an $n$-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve and let X and Y be vector fields tangent to S along $\alpha$. Show that (i) $(X + Y)' = X' + Y'$ ; (ii) $(f X)' = f' X + f X'$ for all smooth functions $f$ along $\alpha$.

13. Find the global parametrization of the plane curve oriented by $\nabla f/\|\nabla f\|$ where $f$ is the function defined by $ax_1 + bx_2 = c, (a, b) \neq (0, 0)$.

14. Let V be a finite dimensional vector space with *dof* product and let $L : V \to V$ be a self-adjoint linear transformation on V. Show that there exists an orthonormal basis for V consisting of eigen vectors of L.

15. Find the Gaussian curvature of the parametrized 2-surface.

    $\phi(t, \theta) = (\cos \theta, \sin \theta, t)$.

16. Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrization of C.

    Prove that $\beta$ is either one-one or periodic. Also show that $\beta$ is periodic if and only C is compact.

$(5 \times 2 = 10)$

## Part C

*Answer any* **three** *questions.*
*Each question has weight* 5.

17. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of $f$ and let $f(p) = c$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at $p$ is equal to $[\nabla f(p)]^{\perp}$.

18. Let S be a compact connected oriented $n$-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Prove that the Gauss map maps S onto the unit sphere $S^n$.

19. Let C be an oriented plane curve. Prove that there exists a global parametrization of C if and only if C is connected.

20. (a) Prove that the Weingarten map $L_p$ is self adjoint.

(b) Prove that $\nabla_v (f X) = (\nabla_v f) \times (p) + P(p)(\nabla_v X)$.

21. (a) Prove that on each compact oriented $n$-surface S in $R^{n+1}$, there exists a point P such that the second fundamental form of P is definite.

(b) Find the Gauss-Kronecker and mean curvatures of $f\left(x_1, x_2 \ldots x_{n+1}\right) = c$ oriented by $\nabla f / \|\nabla f\|$, where $x_1 + x_2 + \ldots + x_{n+1} = 1$, $p = (1, 0 \ldots 0)$.

22. Let S be an $n$-surface in $R^{n+1}$ and let $f : S \to R^k$. Prove that $f$ is smooth if and only if $f \circ \phi : U \to R^k$ is smooth for each local parametrization $\phi : U \to S$.

$(3 \times 5 = 15)$