G 2775 (Pages : 2) Reg. No.................................

Name.....................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, AUGUST 2014

Second Semester

Faculty of Science

Branch I (a)—Mathematics

MTO 2C 06—ABSTRACT ALGEBRA

(2012 admission onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question carries weight 1.

1. Find all subgroups of $Z_2 \times Z_4$ of order 4.
2. How many polynomials are there of degree ≤ 2 in $Z_5(x)$?
3. Show that the polynomial $x^2 + 1$ is irreducible in $Z_3(x)$.
4. Find the degree and a basis for $Q\left(\sqrt{2}, \sqrt{6}\right)$ over $Q\left(\sqrt{3}\right)$.
5. Find two sylow 2-subgroups of S_4 and show that they are conjugate.
6. Find all conjugates in C of $\sqrt{1+\sqrt{2}}$ over Q.
7. What is the order of $G\left(\left(Q\sqrt[3]{2}\right)/Q\right)$?
8. The field $K = Q\left(\sqrt{2}, \sqrt{3}, \sqrt{5}\right)$ is a finite normal extension of Q. Find $|\lambda\left(Q\left(\sqrt{2}+\sqrt{6}\right)\right)$.

$(5 \times 1 = 5)$

Part B

Answer any **five** *questions.*
Each question carries weight 2.

9. Let $(a_1, a_2, \ldots, a_n) \in \pi_{i=1}^{n} Gi$. If a_i is of finite order r_i in G_i, then show that the order of $(a_1, a_2, \ldots, a_n) \in \pi_{i-1}^{n} Gi$ is equal to the least common multiple of all the r_i.
10. Prove that the polynomial $x^2 - 2$ has no zeros in the rational numbers and thus $\sqrt{2}$ is not a rational number.

Turn over

11. Show that a finite extension field E of a field F is an algebraic extension of F.
12. State and prove fundamental theorem of algebra.
13. Let G be finite group. Prove that G is a p-group if and only if $|G|$ is a power of p.
14. If H and K are finite subgroups of G, show that :

$$|HK| = \frac{(|H|)(|K|)}{|H \cap K|}.$$

15. If $E \leq F$ is a splitting field over F, show that every irreducible polynomial in F [x] having a zero in E splits in E.
16. Show that every field of characteristic zero is perfect.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question carries weight 5.

17. Let F be a subfield of a field E, let α be any element of E, and let x be an indeterminate. Prove that the map $\phi_\alpha : F[x] \to E$ defined by $\phi_\alpha\left(a_0 + a_1 x + \ldots a_n x^n\right) = a_0 + a_1 \alpha + \ldots a_n \alpha^n$ is a homeomorphism of F [x] into E. Also show that $\phi_\alpha(a) = a$ for $a \in F$ and $\phi_\alpha(x) = \alpha$.
18. (a) State and prove Eisenstein criterion.

    (b) Show that the polynomial $\phi_p(x) = \dfrac{x^p - 1}{x - 1}$ is irreducible over Q for any prime p.
19. Let F be a field and let $f(x)$ be a non-constant polynomial in F [x]. Prove that there exists an extension field E of F and an $\alpha \in E$ such that $f(\alpha) = 0$.
20. (a) If F is a field of prime characteristic p with algebraic closure $\overline{F}$, then show that $x^{p^n} - x$ has p^n distinct zeros in $\overline{F}$.

    (b) Let p be a prime and let $n \in Z^+$. Show that if E and E' are fields of order p^n, then $E \simeq E'$.
21. Let F be a finite field of characteristic p, then show that the map $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism. Also show that $F(\sigma_p) = Z_p$.
22. Prove that every finite field is perfect.

(3 × 5 = 15)