QP CODE: 22002312

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Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , NOVEMBER 2022

Second Semester

CORE - ME010201 - ADVANCED ABSTRACT ALGEBRA

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

DD4A6580

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define an algebraically closed field. Give an example.
2. *If E is a finite field of characteristic p, then prove that E contains exactly p^n elements for some positive integer n.*
3. Define Unique Factorization Domain and Principal Ideal Domain.
4. Express $2x^2 - 3x + 6$ as a product of its content with a primitive polynomial in $\mathbb{Z}[x]$
5. Define integral domain and multiplicative norm on an integral domain.
6. *State the isomorphism extension theorem. Use this theorem to prove that if $E \leq \overline{F}$ is an algebraic extension of a field F and $\alpha, \beta \in E$ are conjugate over F, then the conjugation isomorphism $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ can be extended to an isomorphism of E onto a subfield of $\overline{F}$.*
7. *What is the order of $G(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})/\mathbb{Q})$?*
8. *Prove that the splitting field over $\mathbb{Q}$ of $x^3 - 1$ is of degree 2 over $\mathbb{Q}$.*
9. *Describe the group of the polynomial $x^3 - 1 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.*
10. *Define symmetric function over a field F.*

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Show that $\mathbb{R}[x]/ < x^2 + 1 >$ is isomorphic to the field $\mathbb{C}$ of complex numbers.

12. *If E is a finite extension field of a field F and K is a finite extension field of E then prove that K is a finite extension of F and [K:F]= [K:E][E:F]*

13. Let D be a UFD and let F be a field of quotients of D. Let f(x) in D[x] has degree greater than 0. If f(x) is irreducible in D[x], then prove that f(x) is also irreducible in F[x]. Also if f(x) is primitive in D[x] and irreducible in F[x], then prove that f(x) is irreducible in D[x].

14. Prove that every Euclidean domain is a UFD.

15. *Let* $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ *and* $F = \mathbb{Q}(\sqrt{2})$. *Find* $G(E/F)$ *and prove that it is isomorphic to the Klein* 4 *- group.*

16. *If* $F \leq E \leq K$, *where* K *is a finite extension field of a field* F, *then prove that* $\{K : F\} = \{K : E\}\{E : F\}$ *. Illustrate this result with an example.*

17. *If* K *is a finite extension of* E *and* E *is a finite extension of* F, *then prove that* K *is separable over* F *if and only if* K *is separable over* E *and* E *is separable over* F.

18. Let K be a finite normal extension of a field F and let E be an extension of F, where $F \leq E \leq K \leq \overline{F}$. Prove the following.
    a) K is a finite normal extension of E.
    b) $G(K/E)$ is precisely the subgroup of $G(K/F)$ consisting of all those automorphisms that leave E fixed.
    c) Two automorphisms σ *and* τ *in* $G(K/F)$ *induce the same isomorphism of* E *onto a subfield of* $\overline{F}$ *if and only if they are in the same left coset of* $G(K/E)$ *in* $G(K/F)$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. a) Prove that the set of all constructible real numbers forms a subfield of the field of real numbers
    b) Prove that doubling the cube is impossible

20. a) Prove that $\mathbb{Z}[i]$ is an integral domain.
    b) Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

21. a) State and prove the Conjugation Isomorphism Theorem.
    b) Prove that complex zeros of polynomials with real coefficients occur in conjugate pairs.

22. Prove the following.
    a) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Then all zeros of $f(x)$ in $\overline{F}$ have the same multiplicity.
    b) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Then $f(x)$ has a factorization in $\overline{F}[x]$ of the form $a\prod_i (x - \alpha_i)^{\nu}$ where α_i are the distinct zeros of $f(x)$ in $\overline{F}$ and $a \in F$.
    c) If E is a finite extension of a field F, then $\{E : F\}$ divides $[E : F]$.

(2×5=10 weightage)