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Reg. No..............................

Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2015

Third Semester

Faculty of Science

Branch I-(A) : Mathematics

MT 03 C13—DIFFERENTIAL GEOMETRY

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any five questions.
Each question has weight 1.

1. Show that the graph of any function $f : R^n \to R$ is a level set for some function $F : R^{n+1} \to R$.

2. Show that the gradient of f at $P \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at P.

3. Define a geodesic, show that the parametrized curve $\alpha(t) = (\cos(at + b), \sin(at + b), (ct + d))$ is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in R^3.

4. Define Gauss Map and Spherical image of an oriented n-surface S.

5. Show that $\nabla_v (X + Y) = \nabla_v X + \nabla_v Y$ for all smooth vector fields X and Y on S.

6. Define a global parametrization of an oriented plane curve C.

7. Find the normal curvature $k(v)$ for each tangent direction v at the point $p = (1, 0 \ldots\ldots 0)$ of $x_1 + x_2 \ldots\ldots + x_{n+1} = 1$ oriented by $\nabla f / \|\nabla f\|$, where

   $$f(x_1, x_2 \ldots\ldots x_{n+1}) = x_1 + x_2 + \ldots\ldots\ldots + x_{n+1}.$$

8. Define a parametrized n-surface in R^{n+k}. Give an example.

$(5 \times 1 = 5)$

Turn over

Part B

*Answer any **five** questions.*
Each question has weight 2.

9. Find and sketch the gradient field of $f(x_1, x_2) = x_1 - x_2^2$.

10. Show that the cylinder $x_1^2 + x_2^2 = 1$ can be represented as a level set of $f(x_1, x_2, x_3) = -x_1^2 - x_2^2$.

11. Describe the spherical image when $n = 1$ and when $n = 2$ of the n surface $x_2^2 + \ldots\ldots x_{n+1}^2 = 1$

    oriented by $\nabla f \|\nabla f\|$ where f is the function $x_2^2 + \ldots\ldots + x_{n+1}^2$.

12. Find the velocity, the acceleration and the speed of $\alpha(t) = (\cos t, \sin t, t)$.

13. Compute $\nabla_v X$ where $V \in R_p^{n+1}$, $X(x_1, x_2) = \left(x_1, x_2, x_1 x_2, x_2^2\right)$, $V = (1, 0, 0, 1)$ and $n = 1$.

14. For each 1-form w on $U \left(U \text{ open in } R^{n+1}\right)$ show that there exist unique functions

    $f_i : U \to R \left(i \in \{1, 2, \ldots\ldots n + 1\}\right)$ such that $w = \sum_{i=1}^{n+1} f_i \, dx_i$.

15. Find the Gaussian curvature $K : S \to R$ where S is the surface given by $x_1^2 + x_2^2 - x_3^2 = 0$, $x_3 > 0$.

16. State and prove inverse function theorem for n-surfaces.

$$(5 \times 2 = 10)$$

Part C

*Answer any **three** questions.*
Each question has weight 5.

17. Let U be an open set in R^{n+1} and let $f : U \to R$ be smooth. Let $p \in U$ be a regular point of f and

    let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $\left[\nabla f(p)\right]^\perp$.

18. Let S be an n-surface in R^{n+1}, let $p \in S$, and let $v \in S_p$. Prove that there exists an open interval

    I containing o and a geodesic $\alpha : I \to S$ such that (i) $\alpha(0) = p$ and $\dot\alpha(0) = v$; (ii) If $\beta : \bar{I} \to S$ is any

    other geodesic in S with $\beta(0) = p$ and $\dot\beta(0) = v$, then $\bar{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \bar{I}$.

19. Prove that the Weingarten map L_p is self-adjoint.

20. Let η be the 1-form on $R^2 - \{0\}$ defined by $\eta = -\dfrac{x^2}{x_1^2 + x_2^2} dx_1 + \dfrac{x_1}{x_1^2 + x_2^2} dx_2$.

   Prove that for $\alpha : \{a, b\} \to R^2 - \{0\}$, any closed piecewise smooth parametrized curve in $R^2 - \{0\}$,

   $\displaystyle\int_\alpha \eta = 2\pi k$ for some integer k.

21. Let $\phi = U \to R^{n+1}$ be a parametrized n-surface in R^{n+1} and let $p \in U$.

   Prove that there exists an open set $U_1 \subset U$ about P such that $\phi(U_1)$ is an n-surface in R^{n+1}.

22. Prove that on each compact oriented n-surface S in R^{n+1} there exists a point p such that the second fundamental form at p is definite.

$(3 \times 5 = 15)$