22002316

QP CODE: 22002316

Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , NOVEMBER 2022

Second Semester

CORE - ME010205 - MEASURE AND INTEGRATION

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

4FD50FF3

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Prove that the Lebesgue outer measure is translation invariant.
2. Define measurability of a set. Prove that any countable set is measurable.
3. Explain the construction of Cantor set.
4. Define Lebesgue integral of a simple function ϕ defined on a set of finite measure E . Assuming that Lebesgue integration is linear, prove that it is monotonic on simple functions.
5. Prove that Riemann integrablity of bounded function f defined on a closed bounded interval $[a, b]$ implies Lebesgue integrablity of f.
6. Define integrablity of a nonnegative Lebesgue measurable function. Prove that if f is Lebesgue integrable over E , then f is finite valued a.e.
7. Let $(X, \mathcal{M}, \mu)$)be a measure space. If $A \subseteq B \subseteq X$ with $\mu(A) = 0$, prove that $\mu(B \sim A) = \mu(B)$.
8. Let $(X, \mathcal{M}, \mu)$ be a general measure space and $\{f_n\}$ be a sequence of measurable functions on X. Prove that $sup\{f_n\}$ and $inf\{f_n\}$ are measurable.
9. *Let $(X, \mathcal{M}, \mu)$ be a measure space and ψ be a nonnegative simple functions on X. If A and B are disjoint measurable subsets of X, then prove that* $\int_{AUB} \psi d\mu = \int_A \psi \, d\mu + \int_B \psi \, d\mu$
10. Define Measure space, Product Measure, Measurable Rectangle and x-section of a function.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let E be any set of real numbers. Then prove that E is measurable if and only if for each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} - E) < \epsilon$

12. If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then prove that $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

13. Define Lebesgue measurability of a function. Prove that a function f on a Lebesgue measurable set E is Lebesgue measurable if and only for each open set O the inverse image $f^{(-1)}(O)$ is Lebesgue measurable.

14. Define pointwise convergence a.e of a sequence $\{f_n\}$of functions. Prove that, if $\{f_n\}$ *i*s sequence of measurable functions on E that converges pointwise a.e. to f, then f is measurable.

15. Let f be a real integrable function on the Lebesgue measure space $(R, \mathcal{M}, m)$. For $E \in \mathcal{M}$ define $\nu(E) = \int_E f dm$. Find the Hahn and Jordan decompositions for ν..

16. Prove that a countable collection of measurable sets is measurable.

17. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence of nonnegative measurable functions on X. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each x in X. Then prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$

18. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ a sequence of non-negative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for all x.in X. Then prove that $\lim_{n\to\infty} \int_X h_n \, d\mu = 0$ if and only if $\{h_n\}$ is uniformly integrable and tight.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19.
   1. Define algebra and σ-algebra.
   2. Prove that the collection of measurable sets is a σ-algebra that contains the σ-algebra of Borel sets

20. Let f and g are bounded Lebesgue measurable functions defined on a set of finite measure E. Prove that
   1) For any α and β, $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$
   2) If $f \leq g$ on *E*, then $\int_E f \leq \int_E g$
   3) *F*or disjoint subsets A and B of E, $\int_{AUB} f = \int_A f + \int_B f$

21. (i) State and prove the Hahn decomposition theorem.

    (ii) Prove that the Hahn decomposition is unique except for null sets.

22. (a) State and prove Radon Nikodym Theorem.

    (b) Explain Radon Nikodym derivative

(2×5=10 weightage)