F 7735 (Pages : 3) Reg. No.................................

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M.Sc. DEGREE (C.S.S.) EXAMINATION, FEBRUARY 2014

First Semester

Faculty of Science

Branch I (A) : Mathematics

MTO IC 03—MEASURE THEORY AND INTEGRATION

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Define outer measure. Show that if $m^*(\mathrm{E}) = 0$, then E is measurable.
2. Show that if E is a measurable set, then each translate $\mathrm{E} + y$ is also measurable.
3. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$$

is not Riemann integrable.

4. Show that we may have strict inequality in Fatou's lemma.
5. Define a measure space.
6. Show that union of a countable collection of positive sets is positive.
7. Show that if $f_n \to f$ a.u, then $f_n \to f$ in measure.
8. If (X, S, μ) and $(Y, \mathfrak{I}, \upsilon)$ are σ-finite measure spaces, then define the product measure $\mu \times \upsilon$ on $S \times \mathfrak{I}$.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. If $\{A_n\}$ is a countable collection of sets of real numbers, then show that $m^*(\cup A_n) \leq \Sigma\, m^*(A_n)$.

10. Let $\mathrm{E} \subset [0, 1)$ be a measurable set. Then show that for each $y \in [0, 1)$, the set $\mathrm{E} \dot{+} y$ is measurable and $m(\mathrm{E} \dot{+} y) = m(\mathrm{E})$. Have $x \dot{+} y$ denote the sum modulo 1 of x and y.

Turn over

11. Let (f_n) be a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each $x \in E$, then show that $\int_E f = \lim \int_E f_n$.

12. Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \leq \int_E |f|$. Does the integrability of $|f|$ imply that of f.

13. State and prove Lebesgue convergence theorem.

14. Show that the set function μ^* defined by $\mu^*(E) = \inf \sum_{i=1}^{\infty} \mu A_i$ where $< A_i >$ ranges over all sequences from a σ algebra such that $E \subset \bigcup_{i=1}^{\infty} A_i$ is an outer measure.

15. Show that if $f_n \to f$ in measure and $g_n \to g$ in measure, then $f_n + g_n \to f + g$ in measure and $\alpha f_n \to \alpha f$ in measure, where α is any real number.

16. If $\mathcal{A}$ is an algebra, then show that the σ-algebra generated by $\mathcal{A}$ is the smallest monotone class containing $\mathcal{A}$.

$(5 \times 2 = 10)$

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. (a) Show that outer measure of an interval is its length.

    (b) Show that the collection $\mathcal{m}$ of measurable sets is a σ-algebra.

18. Let f be defined and bounded on a measurable set E with mE finite. Show that

$$\inf_{f=4} \int_E \psi(x)\,dx = \sup_{f \geq \phi} \int_E \phi(x)\,dx$$ for all simple function ϕ and ψ, if and only if f is measurable.

19. Let f be an increasing real valued function on the interval $[a, b]$. Then show that f is differentiable almost everywhere. Also show that the derivative f' is measurable and $\int_a^b f'(x)\,dx \leq f(b) - f(a)$.

20. (a) Let $(X, \mathscr{B})$ be a measurable space, $<\mu_n>$ a sequence of measures which converge set wise to a measure μ and $<f_n>$ a sequence of non-negative measurable functions which converge pointwise to the function f. Then show that $\int f \, d\mu \le \underline{\lim} \int f_n d\mu_n$.

(b) State and prove Hahn Decomposition theorem.

21. State and prove Radon-Nikodym theorem.

22. (a) If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure. Show that there exists a measurable function f such that $f_n \to f$ in measure.

(b) Let $f_n \to f$ in measure where f and each f_n are measurable functions. Show that there exists a subsequence $\{n_i\}$ such that $fn_i \to f$ a.e.

(3 × 5 = 15)