QP CODE: 22001757

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Reg No : ....

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M Sc DEGREE (CSS) EXAMINATION, AUGUST 2022

Fourth Semester

Core - ME010402 - ANALYTIC NUMBER THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

6FF46B9C

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define Euler Totient function $\phi(n)$. Prove that $\sum_{d|n} \phi(d) = n$ for $n \geq 1$.

2. Prove that $log\ n = \sum_{d|n} \Lambda(d)$ for $n \geq 1$.

3. Define arithmetic mean, average order and big oh notation of arithmetical functions.

4. What is mean by Taubarian theorems ? State Shapiro's Taubarian Theorem.

5. Show that $\forall\ x \geq 1, \sum_{p \leq x} \frac{log\ p}{p} = log\ x + O(1)$.

6. Assume $a \equiv b(mod\ m)$. If $d|m$ and $d|a$ then prove that $d|b$.

7. $\{a_1, a_2, \ldots a_m\}$ is a complete residue system modulo m . Prove that $\{ka_1, ka_2, \ldots ka_m\}$ is a complete residue system modulo m for every k with $(k, m) = 1$.

8. For any prime p , prove that $(p-1)! \equiv -1(mod\ p)$.

9. Write formulae for evaluating $(-1|p)$ and $(2|p)$. Hence find $(2|13)$.

10. Define the exponent of a modulo m. Prove that $exp_m(a)|\phi(m)$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Prove that for every $n \geq 1$, we have $\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if n is a square} \\ 0 & \text{otherwise.} \end{cases}$

    Also show that $\lambda^{-1}(n) = |\mu(n)|$.

12. (a) State and prove the Legendre's identity.

    (b) If $x \geq 2$ prove that $log[x]! = x\ log\ x - x + O(log\ x)$.

13. For $x \geq 2$,prove that $\vartheta(x) = \pi(x)logx - \int_2^x \frac{\pi(t)}{t}dt$ and $\pi(x) = \frac{\vartheta(x)}{logx} + \int_2^x \frac{\vartheta(t)}{t\ \ log^2t}dt$.

14. Prove that $\frac{1}{6}n\,log\,n < P_n < 12(log\ \ n + n\,log\frac{12}{e}), \forall\,n \geq 1$ where P_n is the the n^{th}prime.

15. State and prove Little Fermat theorem.

16. State and prove Chinese remainder theorem .

17. State and prove Euler's criterion. Hence evaluate $(22|11)$.

18. Using mathematical induction prove that there exists an integer $m < \phi(2^\alpha)$ with $x^m \equiv 1\,mod\,2^\alpha$ when $\alpha \geq 3$ and x is odd.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Prove that (a) two lattice points (a, b) and $(m, n)$are mutually visible if and only if $a - m$and $b - n$ are relatively prime.
(b) a lattice point chosen at random has the probability $\frac{6}{\pi^2}$ of being visible from the origin.

20. Prove that the following relations are logically equivalent.
(a) $\lim\limits_{x\to\infty} \frac{\pi(x)log\ \ x}{x} = 1.$
(b) $\lim\limits_{x\to\infty} \frac{\pi(x)\ log\ \pi(x)}{x} = 1.$
(c) $\lim\limits_{n\to\infty} \frac{P_n}{n\ log\ n} = 1.$

21. (a) Let f be a polynomial with integer coefficients, let $m_1, \ldots, m_r$be positive integers relatively prime in pairs, and let $m = m_1 m_2 \ldots m_r$. Prove that the congruence$f(x) \equiv 0(mod\,m)$ has a solution if and only if each of the congruences $f(x) \equiv o(mod\,m_i) \quad (i = 1, \ldots, r)$ has a solution. Also show that if $v(m)$ and $v(m_i)$denote the number of solutions of $f(x) \equiv 0(mod\,m)$ and $f(x) \equiv o(mod\,m_i)$ for $i = 1, \ldots, r$, respectively, then $v(m) = v(m_1)v(m_2)\ldots v(m_r)$.

(b) Prove that the set of lattice points in the plane visible from the origin contains arbitrarily large square gaps.

22. If p and q are distinct odd primes prove that $(p|q) = \begin{cases} -(q|p) & \text{if } p \equiv q \equiv 3\,mod\,4 \\ (q|p) & \text{if otherwise} \end{cases}$

(2×5=10 weightage)