G 2777 (Pages : 3) Reg. No..............................

Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION AUGUST 2014

Second Semester

Faculty of Science

Branch I (A)—Mathematics

MT 02 C08—ADVANCED COMPLEX ANALYSIS

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Define a power series. Expand $\frac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence ?

2. Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$.

3. Define genus and order of an entire function. What is the relation between them ?

4. Show that the ζ-function can be extended to a meromorphic function in the whole plane whose only pole is a simple pole at $s = 1$ with residue 1.

5. State Schwarz-Christoffel formula.

6. Show that a continuous real valued function $u(z)$, which satisfies the mean value property is harmonic.

7. Show that an elliptic function without poles is a constant.

8. Define the term sheaf over an open set D in the complex plane.

$(5 \times 1 = 5)$

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. If $\sum_{n=0}^{\infty} a_n$ converges, then show that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ tends to $f(1)$ as z approaches 1 in such a way that $|1-z|/(1-|z|)$ remains bounded.

Turn over

10. Prove that the infinite product $\prod_{1}^{\infty}(1+a_n)$ with $1+a_n \neq 0$ converges simultaneously with the series $\sum_{1}^{\infty} \log(1+a_n)$ whose terms represent the values of the principal branch of the logarithm.

11. State and prove Jensen's formula.

12. Derive the functional equation of the Riemann Zeta function.

13. Prove that a continuous function $v(z)$ is subharmonic in Ω if and only if it satisfies the inequality

$$v(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta \text{ for every disk } |z - z_0| \leq r \text{ contained in } \Omega.$$

14. Let f be a topological mapping of a region Ω onto a region Ω^1. If $\{z_n\}$ or $z(t)$ tends to the boundary of Ω, then show that $\{f(z_n)\}$ or $f(z(t))$ tends to the boundary of Ω^1.

15. Prove that a non-constant elliptic function has equally many poles as it has zeros.

16. Prove that the Weierstrass $\wp$ function $\wp(z) = \frac{1}{z^2} + \sum_{w \neq 0}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. Obtain the Laurent series expansion of a function which is analytic in the annulus $R_1 < |z - a| < R_2$.

18. Prove that a family $\mathcal{F}$ is normal if and only if its closure $\mathcal{F}$-with respect to the distance function

$$\rho(f, g) = \sum_{k=1}^{\infty} \sqrt{k}\,(f, g) 2^{-k} \text{ is compact.}$$

19. State and prove Riemann mapping theorem.

20. State and prove Harnack's principle.

21. Show that any even elliptic function with periods w_1, w_2 can be expressed in the form $c\prod_{k=1}^{n}\frac{\wp(z)-\wp(a_k)}{\wp(z)-\wp(b_k)}$, provided 0 is neither a zero nor a pole. What is the corresponding form if the function either vanishes or becomes infinite at the origin.

22. Prove that any two bases of the same module are connected by unimodular transformation. Also show that there exists a basis (w_1, w_2) such that $T = w_2/w_1$ satisfies the conditions (i) $\operatorname{Im} T > 0$; (ii) $\frac{-1}{2} < \operatorname{Re} T \le \frac{1}{2}$, (iii) $|T| \ge 1$; (iv) $\operatorname{Re} T \ge 0$ if $|T| = 1$.

(3 × 5 = 15)