F 5112 (Pages : 2) Reg. No.....................

Name.....................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY/FEBRUARY 2017

First Semester

Faculty of Science

Branch I (a)—Mathematics

MT 01 C03—MEASURE THEORY AND INTEGRATION

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question carries 1 *weight.*

1. Define a measurable set. If M*(EI = 0, show that E is measurable.
2. Show that $X_{A \cup B} = X_A + X_B - X_A \cdot X_B$.
3. Show that if $f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$ then $\mathrm{R}\overline{\int_a^b} f(x)\,dx = b - a$ and $\mathrm{R}\underline{\int_a^b} f(x)\,dx = 0$.
4. Show that we may have strict inequality in Fatou's lemma.
5. Define a measurable space and a measure μ on a measurable space.
6. Show that the union of a countable collection of positive sets is positive.
7. Show that if $f_n \to f$ a.u., then $f_n \to f$ a.e.
8. If $[X, S, \mu]$ and $[Y, y, \upsilon]$ be σ-finite measure spaces, define the product measure $\mu \times \upsilon$ on $S \times T$.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question carries 2 *weight.*

9. Show that the interval (a, ∞) is measurable.
10. Give an example of a non-measurable set. Justify your answer.
11. State and prove bounded convergence theorem.
12. State and prove Lebesgue convergence theorem.
13. If If $Ei \in B$, $\mu E_1 < \infty$ and $E_i \supset E_{i+1}$, show that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu E_n$.

Turn over

14. Show by an example that Hahn decomposition need not be unique.

15. Let $\{f_n\}$ be a sequence of non-negative measurable functions and let f be a measurable function such that $f_n \to f$ is measure. Show that $\int f d\mu \leq \liminf \int f_n d\mu$.

16. Prove that the class of elementary sets is an algebra.

$(5 \times 2 = 10)$

Part C

Answer any **three** *questions.*
Each question carries 5 weight.

17. (a) Show that outer measure of an interval is its length.
    (b) Prove that every Borel set is measurable.

18. (a) State and prove Monotone convergence theorem.
    (b) Show that monotone convergence theorem need not hold for a decreasing sequence of functions.
    (c) Let f be a non-negative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e.

19. Let f be an increasing real-valued function on the interval $[a, b]$. Prove that f is differentiable a.e., the derivative f^1 is measurable and $\int_a^b f'(a)\, dx \leq f(b) - f(a)$.

20. (a) State and prove Hahn decomposition theorem.
    (b) Show that Hahn decomposition is unique except for null sets.

21. Prove that the calss B of μ^* measurable sets is a σ-algebra. Also prove that if $\bar{\mu}$ is μ^* restricted to B, then $\bar{\mu}$ is a complete measure on B.

22. Let $[X, S, \mu]$ and $[Y, \mathfrak{I}, \upsilon]$ be σ-finite measure spaces. For $V \in S \times \mathfrak{I}$ write $\phi(x) = \upsilon(V_x)$, $\psi(y) = \mu(V^y)$ for each $a \in X, y \in Y$. Prove that ϕ is S-measurable, ψ is $\mathfrak{I}$ measurable and $\int_X \phi d\mu = \int_Y \psi d\upsilon$.

$(3 \times 5 = 15)$