QP CODE: 24018056

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, APRIL 2024

Fourth Semester

Core - ME010401 - SPECTRAL THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

F2F19AB0

Time: 3 Hours Weightage: 30

Instructions: (Applicable for **Private Registration, 2020 Admission Onwards**) This question paper contains two sections. Answer section I questions in the answer book provided. Section II Internal examination questions must be answered in the question paper itself. Follow the detailed instructions given under section II.

SECTION I

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. If a normed space X is reflexive , then prove that it is complete.
2. Show that uniform operator convergence $T_n \to T, T_n \in B(X,Y)$, implies strong operator convergence with the same limit.
3. Define fixed point of a mapping. Give an example.
4. Define eigenvalues and eigenvectors of a linear operator $T : D(T) \to X$, where $X \neq \{0\}$ is a complex normed space and $D(T) \subset X$.
5. When we can say that an operator function $S : \Lambda \to B(X,X)$, where Λ be an open subset of C and X is a Banach space is locally holomorphic?
6. Define inverse of an element $x \in A$, where A is an algebra with identity. Show that inverse of an element if it exist is unique.
7. Let A be a complex Banach algebra with identity. Then show that the spectrum $\sigma(x)$ of an $x \in A$ is closed.
8. If X is a finite dimensional normed space then show that the identity operator $I : X \to X$ is not compact.
9. Let P_1 and P_2 be projections of a Hilbert space H onto Y_1 and Y_2 respectively and $P_1P_2 = P_2P_1$. Show that $P_1 + P_2 - P_1P_2$ is a projection of H onto $Y_1 + Y_2$.

10. Let P_1 and P_2 be projections defined on a Hilbert space H and let $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$. If the difference $P = P_2 - P_1$ is a projection, prove that $Y_1 \subset Y_2$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let (x_n) be a weakly convergent sequence in a normed space X, say, $x_n \rightharpoonup x$. Then prove the following,

(a) The weak limit x of (x_n) is unique.

(b) Every subsequence of (x_n) converges weakly to x.

(c) The sequence $(\|x_n\|)$ is bounded.

12. Let X and Y are normed spaces. Prove that $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ defines a norm on $X \times Y$.

13. Prove that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is closed.

14. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

15. If T is a compact linear operator on a normed space X, Prove that for every $\lambda \neq 0$, dim $\mathcal{N}(T_\lambda^n) < \infty$ and range of T_λ^n is closed.

16. Prove that every normed space X can be expressed as the direct sum of two closed subspaces, which are the null space and range of the operator T_λ^r where $T : X \to X$ is a compact linear operator and $\lambda \neq 0$.

17. Let $T : H \to H$ be a bounded linear operator on a complex Hilbert space H. Then prove that a number $\lambda \in \rho(T)$ if and only if there exists a $c > 0$ such that $\|T_\lambda x\| \geq c\|x\| \quad \forall x \in H$.

18. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Prove that $\sup\limits_{\|x\|=1} \langle Tx, x\rangle \in \sigma(T)$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. State and prove Bounded Inverse Theorem.

20. State and prove Spectral Mapping Theorem for Polynomials.

21. If B is a subset of a metric space X , then prove the following

a. If B is relatively compact, then B is totally bounded.

b. If B is totally bounded and X is complete, then B is relatively compact.

c. If B is totally bounded, then for every $\epsilon > 0$ B has a finite ϵ - net contained in B.

d. If B is totally bounded, then B is separable.

22. Let (T_n) be a sequence of bounded self-adjoint linear operators on a complex Hilbert space H such that $T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots \leq K$ where K is a bounded self-adjoint linear operator on H. Suppose that any T_j commutes with K and with every T_m. Then prove that (T_n) is strongly operator convergent and the limit operator T is linear, bounded and self-adjoint and satisfies $T \leq K$.

(2×5=10 weightage)