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Reg. No.................................

Name.....................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2017

Third Semester

Faculty of Science

Branch I (A) : Mathematics

MT 03 C12—FUNCTIONAL ANALYSIS

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Define a Schauder basis for a normed space X. Give an example.

2. Give examples of subspaces of l^∞ and l^2 which are not closed.

3. Define the dual space of a normed space. What is the dual space of R^n.

4. Define an inner product space. Show that if $x \perp y$ in an inner product space X, then $\| x+y \|^2 = \| x \|^2 + \| y \|^2$.

5. Give an example of $x \in l^2$ such that we have strict inequality occurs in Bessel inequality.

6. Show that for any bounded liner operator T on a Hilbert space H, the operator $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$ are self adjoint, where T* denotes the Hilbert adjoint operator of T.

7. Define a reflexive space. Give example of a space which is :
   (i) Reflexive.
   (ii) Non-reflexive.

8. Of what category is the set of all rational numbers (a) in R ; (b) in itself. Also of what category the set of all integers in R.

(5 × 1 = 5)

Turn over

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Prove that on a finite dimensional Vector space X any two norms are equivalent.

10. State and prove Riesz's lemma.

11. Let X and Y be metric spaces and $T : X \to Y$ be a continuous mapping. Prove that the image of a compact subset M of X under T is compact.

12. Show that a finite dimensional vector space is algebraically reflexive.

13. Let μ be a subset of and inner product space X. Prove the following :

    (a) $x \perp \mu \Rightarrow x = 0$.

    (b) If X is complete, the $x \perp \mu \Rightarrow x = 0$ is sufficient for the totality of μ in X.

14. Prove that in every Hilbert space $H \neq \{0\}$ there exists a total orthonormal set.

15. Let $S, T \in B(X, Y)$, show that $(S + T)^{x} = S^{x} + T^{x}$ and $(\alpha T)^{x} + \alpha T^{x}$.

16. Prove that if the dual space X' of a normed space X is separable, then X itself is separable.

$(5 \times 2 = 10)$

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. (i) Let T be a liner operator. Prove the following :

    (a) The range $\mathcal{R}(\mathcal{T})$ is a vector space.

    (b) If dim $\mathcal{D}(\mathcal{T}) = n < \infty$, then dim $\mathcal{R}(\mathcal{T}) \leq n$.

    (c) The null space $\mathcal{N}(\mathcal{T})$ is a vector space.

    (ii) Let T be a bounded linear operator. Prove the following :

    (d) $x_n \to x$ implies $Tx_n \to T_x$ where $x_n, x \in \mathcal{D}(\mathcal{T})$.

    (e) the null space $\mathcal{N}(\mathcal{T})$ is closed.

18. Prove that the dual of l^p is l^q, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

19. (i) State and prove Schwarz inequality and triangle inequality.

    (ii) show that in an inner product space, if $x_n \to x$ and $y_n \to y$, then $< x_n, y_n > \to < x, y >$.

20. (i) State and prove Bessel inequality.

(ii) Describe Gram-Schmidt process for orthonormalizing a linearly independent sequence in an inner product space.

21. Let H be a Hilbert space, prove that :

(a) If H is separable, then every orthonormal set in H is countable.

(b) If H contains an orthonormal sequence which is total in H, then H is separable.

22. State and prove Hahu-Banach extension theorem for linear functionals.

(3 × 5 = 15)