QP CODE: 23002640

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, MARCH 2023

Third Semester

Faculty of Science

CORE - ME010304 - FUNCTIONAL ANALYSIS

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

936E48B3

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. *Show that the discrete metric space is complete.*
2. When do you say that a subset M of a metric space X is compact. Give an example of a compact metric space.
3. Prove that the inverse of a linear operator T exists if and only if null space of T is equal to $\{0\}$.
4. Define a bounded linear operator on a normed space. Give an example for a bounded linear operator.
5. Let Tbe a bounded linear operator. Then prove that, if $x_n \to x$, where $x_n, x \in D(T)$ implies $Tx_n \to Tx$
6. Define a Hilbert space.
7. Define orthonormal sets and orthonormal sequences.
8. Define isomorphisc Hilbert spaces.
9. *Define partially ordered set. Give an example.*
10. *Let X and Y are normed spaces and $S,T \in B(X,Y)$ then prove that $(S+T)^{\times} = S^{\times} + T^{\times}$ and $(\alpha T)^{\times} = \alpha T^{\times}$ where α be any scalar.*

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Define a normed space. Prove that norm is a continuous function.

12. Prove that on a finite dimensional vector space X, any two norms are equivalent.

13. Let $f : C[a, b] \to R$ be a function defined by $f(x) = \int_a^b x(t)dt$. Prove that f is a bounded linear functional on $C[a, b]$.

14. If X is a finite dimensional normed space, then prove that X^* is also finite dimensional and $dim X = dim X^*$.

15. Prove that for any subset $M \neq \phi$ of a Hilbert space H, the span of M is dense in H if and only if $M^\perp = \{0\}$.

16. Let e_k be an orthonormal sequence in a Hilbert space H.

a) Prove that the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges in the norm on H if and only if the series $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.

b) Prove that for any $x \in H$, the series $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges in the norm of H.

17. *Define the Hilbert-adjoint operator of a bounded linear operator $T : H_1 \to H_2$ where H_1 and H_2 are Hilbert spaces. Prove that the Hilbert-adjoint operator T^* of T exists, is unique and is a bounded linear operator with norm $||T^*|| = ||T||$.*

18. *Define self -adjoint linear operator. Let T be a bounded linear operator on a complex Hilbert space H, prove that T is self-adjoint if and only if $< Tx, x >$ is real for all $x \in H$.*

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (i) A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.
(ii) In l^∞, let Y be the subset of all sequences with only finitely many non zero terms. Show that Y is a subspace of l^∞ but not a closed subspace.

20. Prove that the set of all bounded linear operators from a Normed space X into a Normed space Y is a Normed space. Hence a Banach space if Y is a Banach space.

21. a) State and prove Riesz's Theorem.
b) Show that any linear functional f on $\mathbf{R}^3$ can be represented by a dot product,
$f(x) = x \cdot z = \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3$.

22. State and prove Hahn-Banach theorem for complex vector spaces.

(2×5=10 weightage)