**QP CODE: 22001758**

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# M Sc DEGREE (CSS) EXAMINATION, AUGUST 2022

## Fourth Semester

## Elective - ME800401 - DIFFERENTIAL GEOMETRY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

4F390783

Time: 3 Hours

Weightage: 30

**Part A (Short Answer Questions)**

*Answer any **eight** questions.*

*Weight **1** each.*

1. Find the level set of the function $f(x_1, x_2) = x_1^2 + x_2^2$ for $c = -2, 0, 2, 5$
2. Define an $n$-surface. Give an example.
3. Write a note on Gauss Map and sketch it for 1-surface in $\mathbb{R}^2$ .
4. Describe the spherical image, when $n = 1$, of $x_1^2 - x_2^2 - \ldots - x_{n+1}^2 = 4, x_1 > 0$ oriented by $\mathbf{N} = \frac{-\nabla f}{||\nabla f||}$.
5. Define parallel transport and prove that it is a linear map.
6. Define covariant derivative of the tangent vector field $\mathbf{X}$ with respect to $\mathbf{v} \in S_p$. Show that $D_{\mathbf{v}}(f\mathbf{X}) = (D_{\mathbf{v}}f)\mathbf{X}(p) + f(p)D_{\mathbf{v}}\mathbf{X}$, for all smooth tangent vector fields $\mathbf{X}$ and $\mathbf{Y}$ on $S$ and all smooth functions $f : S \to R$.
7. Explain radius of curvature of a plane curve at the point p with definitions of circle of curvature and center of curvature.
8. Define differential of a smooth function. If $f$ is a smooth function on an open set $U \subset \mathbb{R}^{n+1}$ and $h : \mathbb{R} \to \mathbb{R}$ is smooth, show that $d(h \circ f) = (h' \circ f)df$.
9. *Define the first fundamental form and the second fundamental form of an oriented n-surface in $\mathbb{R}^{n+1}$ at a point.*
10. Define the following:
    a) tangent bundle of an open set in $\mathbb{R}^n$
    b) tangent bundle of an n-surface S in $\mathbb{R}^{n+1}$
    c) differential of a smooth map $\varphi : S \to \mathbb{R}^m$.

(8×1=8 weightage)

## Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let $U$ be an open set in $\mathbb{R}^{n+1}$ and let $\mathbf{X}$ be a smooth vector field on $U$. Suppose $\alpha : I \to U$ is an integral curve of $\mathbf{X}$ with $\alpha(0) = \alpha(t_0)$ for some $t_0 \in I, t_0 \neq 0$ . Show that $\alpha$ is periodic

12. Let $f : U \to \mathbb{R}$ be a smooth function where $U \subseteq \mathbb{R}^{n+1}$ is an open set and $\alpha : I \to U$ be a parametrized curve. Show that $fo\alpha$ is a constant if and only if $\alpha$ is everywhere orthogonal to the gradient of $f$.

13. If a parametrized curve $\alpha$ in the unit $n$-sphere $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$ is of the form $\alpha(t) = (\cos at)e_1 + (\sin at)e_2$ for some pair of orthogonal unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and $a \in \mathbb{R}$ then show that it is a geodesic. Is the converse true? Justify.

14. A parametrized curve $\alpha : I \to S$ is a geodesic if and only if its covariant acceleration $(\dot{\alpha})'$ is zero along $\alpha$.

15. a) Compute $\nabla_{\mathbf{v}}\mathbf{X}$ where $\mathbf{v} \in \mathbb{R}^{n+1}_p, p \in \mathbb{R}^{n+1}$ and $\mathbf{X}$ is given by $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1), \mathbf{v} = (cos\theta, sin\theta, -sin\theta, cos\theta), n = 1$.
    b) Suppose $\mathbf{X}$ is a smooth unit vector field on an $n$-surface $S$ in $\mathbb{R}^{n+1}$. Show that $\nabla_{\mathbf{v}}\mathbf{X} \perp \mathbf{X}(p)$ for all $\mathbf{v} \in S_p, p \in S$. Show further that if $\mathbf{X}$ is a unit tangent vector field on $S$, then $D_{\mathbf{v}}\mathbf{X} \perp \mathbf{X}(p)$.

16. a) What do you mean by reparametrization of a parametrized curve.
    b) Show that local parametrizations of plane curves are unique upto reprametrization.

17. *Find the Gaussian curvature of the hyperboloid* $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$.

18. Find the Gaussian curvature of the parametrized 2-surface $\varphi(t, \theta) = (\cos\theta, \sin\theta, t)$.

(6×2=12 weightage)

## Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Let $S$ be an $n$-surface in $\mathbb{R}^{n+1}$, let $\mathbf{X}$ be a smooth tangent vector field on $S$ and let $p \in S$. Show that there exists an open interval $I$ containing 0 and a parametrized curve $\alpha : I \to S$ such that $(i)$ $\alpha(0) = p$ $(ii)$ $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ for all $t \in I$ and $(iii)$ If $\beta : \tilde{I} \to S$ is any other parametrized curve in $S$ satisfying $(i)$ and $(ii)$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$

20. Let $S$ denote the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$. Show that the parametrized curve $\alpha$is a geodesic of $S$ if and only if $\alpha$ is of the form $\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$ for some $a, b, c, d \in \mathbb{R}$.

21. Prove the existence of a global parametrization of any connected oriented plane curve.

22. *Let S be an oriented n-surface in* $\mathbb{R}^{n+1}$ *and let* $\mathbf{v}$ *be a unit vector in* $S_p$, $p \in S$. *Prove that there exists an open set* $V \subset \mathbb{R}^{n+1}$ *containing p such that* $S \cap \mathcal{N}(\mathbf{v}) \cap V$ *is a plane curve. Moreover, the curvature at p of this curve (suitably oriented) is equal to the normal curvature* $k(\mathbf{v})$.

(2×5=10 weightage)